Introduction

For suggested solution to problem 4, last year's suggested solutions by Tsz-Ning Wong were used who I think used suggested solutions by Julia Salmi from three years ago.

Problems and suggested solutions

In problem 1, we investigate what predictions about the play of the game we can make based solely on the knowledge that one of the player's is rational. We also study how the quality of such predictions varies with risk attitude of the player.

Problem 1 (Risk-aversion and implications of rationality)**.**

Consider the following static game:

1. *Set of players,* $\mathcal{I} = \{1, 2\}$

2. *Pure strategy space for player* 1*,* $S_1 = \{t, m, b\}$

3. *Pure strategy space for player* 2*,* $S_2 = \{l, r\}$

4. *An outcome function maps a strategy profile to monetary payments for player* 1 *and player* 2*:* $g : S_1 \times S_2 \to \mathbb{R}^2$*. Player* 1*'s component of the outcome function,* $g^1 : S_1 \times S_2 \to \mathbb{R}$ *is presented in table 1 below*

Further, we assume that player 1 *is rational. Our purpose here is to investigate implication of player* 1*'s rationality on her play, and how the implications vary with her risk attitude.* [1]

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[1]**Reminder**: if we know that a player is rational, all we could say about her play is that she is not going to play a strategy which is never best response. Recall, that a strategy s_1 of player 1 is never best response if there is no conjecture, $\mu \in \Delta(S_2)$, player 1 could make on the actions of player 2, such that s_1 is a best-response to μ.

	l	r
t	3	0
m	1	1
b	0	3

Table 1: Monetary payment for player 1 as a function of the strategy profile

a) Assume that player 1 is risk-neutral and self-centered, that is her von Neumann-Morgenstern utility function, $v : \mathbb{R}^2 \to \mathbb{R}$, is defined by $v(x, y) = x$, where the first argument of the function v is monetary payment to player 1, and the second argument is monetary payment to player 2. Which pure strategies of player 1 are never best response? In other words, with the assumptions we made on player 1's preferences over monetary outcomes and lotteries over them, play of which strategies contradicts her rationality?

Solution. The set of all possible conjectures on the actions of player 2 is $\Delta(S_2)$. Every element in this set, $\mu \in \Delta(S_2)$, can be represented by a number $\pi \in [0, 1]$, which is probability of player 2 taking action l. Consider the following partition of $\Delta(S_2)$: $C_1 := \{\pi : \pi < \frac{1}{2}\}$, $C_2 := \{\pi : \pi > \frac{1}{2}\}$ and $C_3 := \{\pi : \pi = \frac{1}{2}\}$.[2] $\{C_1, C_2, C_3\}$ indeed constitutes a partition of $\Delta(S_2)$, since $C_1 \cup C_2 \cup C_3 = \Delta(S_2)$, $C_1 \cap C_2 = \emptyset$, $C_1 \cap C_3 = \emptyset$, and $C_2 \cap C_3 = \emptyset$.

Now any conjecture on the actions of player 2, player 1 might entertain, must fall into one and only one of the sets among C_1, C_2, C_3. Suppose, the conjecture is in the set C_1. Then, the best response of player 1 to any such conjecture $\pi \in C_1$ is b. To see this, below we write down player 1's expected utility when her conjecture is $\pi \in C_1$ and she choses t, m and b respectively.

Her expected utility when choosing t is

$$\mathbb{E}_{\pi \in C_1}[u(t)] = \pi * 3 + (1 - \pi) * 0 < \frac{3}{2}$$

Her expected utility when choosing m is

$$\mathbb{E}_{\pi \in C_1}[u(m)] = \pi * 1 + (1 - \pi) * 1 = 1$$

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[2]Reminder: according to Wikipedia, "partition of a set is a grouping of the set's elements into non-empty subsets, in such a way that every element is included in one and only one of the subsets"

And eventually, her expected utility when choosing b is

$$\mathbb{E}_{\pi \in C_1}[u(b)] = \pi * 0 + (1 - \pi) * 3 > \frac{3}{2}$$

Thus, for any conjecture $\pi \in C_1$, b maximizes player 1's expected utility, and therefore b is unique best-response to any conjecture $\pi \in C_1$.

Similar calculations would show that for any conjecture $\pi \in C_2$, t maximizes player 1's expected utility, and therefore t is unique best-response to any conjecture $\pi \in C_2$.

For the conjecture $\pi \in C_3$, both b and t yield expected utility of $\frac{3}{2}$ which is larger than expected utility of m - which is still 1 regardless of the conjecture - and therefore b, t, as well as any mixed-strategy which assigns positive probability only to actions b and t constitute best response to such hypothesis.

Thus, we listed all the possible conjectures and found that m is the only pure strategy of player 1, which is never best response. To summarize, we started off with assuming that player 1 is rational, risk-neutral, cares only about his own monetary payment, and arrived at the conclusion that such player would never choose strategy m. That is, with the assumptions we made, all we can say about the outcome of the play is that it must be in the set: $O := \{(t, l), (t, r), (b, l), (b, r)\}$, and we have gotten this far without making any assumptions on rationality or preferences of player 2.

b) Assume that player 1 is risk-averse and self-centered, with her von Neumann-Morgenstern utility function, $v : \mathbb{R}^2 \to \mathbb{R}$, defined by $v(x, y) = \sqrt{x}$, where the first argument of the function v is monetary payment to player 1, and the second argument is monetary payment to player 2. Which pure strategies of player 1 are never best response? In other words, with the assumptions we made on player 1's preferences over monetary outcomes and lotteries over them, play of which strategies contradicts her rationality?

Solution: First, observe that player 1's best response to the conjecture which assigns probability 1 to player 2's strategy l is to play t; similarly, her best response to the conjecture which assigns probability 1 to r is b. Next, suppose that player 1 believes that 2 is equally likely to play l and r. With such conjecture, expected utility from playing t is

$$\mathbb{E}_{\pi=\frac{1}{2}}[u(t)] = \frac{1}{2}\sqrt{3} \approx 0.87$$

Expected utility from playing m is

$$\mathbb{E}_{\pi=\frac{1}{2}}[u(t)] = \frac{1}{2}1 + \frac{1}{2}1 = 1$$

and expected utility from playing b is

$$\mathbb{E}_{\pi=\frac{1}{2}}[u(t)] = \frac{1}{2}\sqrt{3} \approx 0.87$$

Hence, if player 1's subjective belief is such that he thinks that player 2 is equally likely to play l and r, then player 1 is going to play m herself.

Thus, for each pure strategy of player 1, we found a conjecture such that the strategy is a best response to the conjecture. Hence, none of the player 1's strategies are never best-response. Summarizing, we started off with assuming that player 1 is rational, risk-averse, cares only about his monetary payment, and arrived at the conclusion that such player could choose any strategy depending on the conjecture she entertains. That is, with the assumptions we made, we cannot say anything about the outcome of the play.

c) Compare your answers to parts *a)* and *b)*. As player 1 becomes risk-averse, does the set of justifiable strategies of player 1 expands, shrinks or does not change? [3] Does the set of the outcomes of the play consistent with rationality of player 1 expands, shrinks, or does not change as she becomes more risk-averse? What is the intuition behind that? Do you think it is a general result or it is specific to the game we analyzed, and why?

Solution. The set of justifiable strategies expands as player 1 becomes risk-averse: the set of justifiable actions for the risk-neutral player in part *a)* is $\{t, b\}$ while it is $\{t, m, b\}$ for the player who is risk-averse in part *b)*. This is a general result from Weinstein (2015). The intuition behind it is that risk-aversion might make "safe" strategies like $\{m\}$ which provide the same payoff regardless of 2's play justified by nondeterministic conjectures of 1 on 2's play; at the same time, "risky" strategies like $\{t\}$ and $\{b\}$ are best justi-

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[3] A strategy of player 1, s_1, is justifiable if it is not never best response

fied by deterministic conjectures which bear no risk for the decision-maker and therefore degree of risk-aversion does not matter for their justification.

Just knowing that players are rational often does not let us make any predictions on the play apart from "anything can happen". Therefore game theorists typically assume that players are not only rational, but there is also common knowledge of rationality between the players. The difference is that a rational player never plays a strategy which is not a best reply against *any* conjecture on the others' play while a player in a game with common knowledge of rationality never plays a strategy which is not a best reply to *any reasonable* conjecture. In problem 2, we investigate implications of assuming common knowledge of rationality rather than "only" rationality, and clarify what makes a conjecture unreasonable in the view of common knowledge of rationality.[4]

Problem 2 (Common knowledge of rationality and its implications).

Consider the game $<\{1,2\},(S_i),(u_i)>$*, where* $S_i := \mathbb{R}^+$ *and* $u_i(s_1,s_2) = s_i *$ $\max\{0, \frac{\alpha}{\beta} - \frac{s_1}{2\beta} - \frac{s_2}{2\beta}\} - \frac{s_i^2}{2m}$*, where* $\alpha > 0$*,* $\beta > 0$*, and* $m > 0$*. Notice that this captures Cournot duopoly game, with the demand function* $D(p) = 2\max\{0, \alpha - \beta p\}$ *and the cost function* $C(s_i) = \frac{s_i^2}{2m}$*. What are implications of assuming common knowledge of rationality?*[5]

Solution. For player i, there is no conjecture on the action of player $j \neq i$ that would justify (see footnote 3) $s_i > \bar{s}_1 := \alpha\frac{m}{\beta+m}$. That is, rationality implies that player i never chooses $s_i > \bar{s}_1$. To see this, first treat s_j as a parameter of the problem, and write down necessary condition for the optimality of s_i:

$$s_i = \alpha\frac{m}{m+\beta} - \frac{s_j}{2}\frac{m}{m+\beta} \tag{1}$$

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[4]Also see extra-problem 1 where we informally attempt to clarify the concept of common knowledge and contrast it with the concept of mutual knowledge

[5]Hint: assuming common knowledge of rationality allows us to iteratively delete strictly dominated strategies

We can think of (1) as a best response function of player i to a deterministic conjecture assigning probability 1 to j playing s_j. Clearly, s_i is decreasing in s_j. That is, the largest s_i rational i would ever choose is in response to conjecture $s_j = 0$ (since rules of the game dictate that $s_j \geqslant 0$ no player who knows rules of the game would conjecture $s_j < 0$). Thus, we substitute $s_j = 0$ into equation (1), and infer that all the strategies $s_i > \bar{s}_1$ are never best response.

To summarize, we found that rational players would choose strategies from the interval $[0, \alpha \frac{m}{\beta+m}]$. Therefore mutual knowledge of rationality (player i knows that player j is rational) rules out "unreasonable" conjectures which assign positive probability to j's play of the strategies outside of the above interval. Then, there is no reasonable conjecture which would justify $s_i < \underline{s}_1 := \alpha \frac{m}{\beta+m} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2}$ - we get here by substituting the largest possible quantity consistent with the rationality of i's opponent into (1). That is, if i knows that j is rational i would choose strategies from the interval $[\alpha \frac{m}{\beta+m} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2}, \alpha \frac{m}{\beta+m}]$.

Next, assume that player i knows that j knows that i is rational. Then i knows that j chooses a strategy from the interval $[\alpha \frac{m}{\beta+m} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2}, \alpha \frac{m}{\beta+m}]$, and therefore any conjecture which assigns positive probability to j's play of the strategies outside this interval is unreasonable. Now notice that the reasonable conjecture which leads to the largest s_i is $\alpha \frac{m}{\beta+m} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2}$. That tells us that if i knows that j knows that i is rational, i will not choose strategy s_i such that $s_i > \bar{s}_2 := \alpha \frac{m}{\beta+m} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2} + \frac{\alpha}{4} \frac{m^3}{(m+\beta)^3}$

Continuing in the same fashion, we get that

$$\bar{s}_k = \alpha \frac{m}{\beta+m} \sum_{i=0}^{k-1} \left(\frac{m}{2(m+\beta)} \right)^{2i} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2} \sum_{i=0}^{k-2} \left(\frac{m}{2(\beta+m)} \right)^{2i}$$

and:

$$\underline{s}_k = \alpha \frac{m}{\beta+m} \sum_{i=0}^{k-1} \left(\frac{m}{2(m+\beta)} \right)^{2i} - \frac{\alpha}{2} \frac{m^2}{(\beta+m)^2} \sum_{i=0}^{k-1} \left(\frac{m}{2(\beta+m)} \right)^{2i}$$

Assuming common knowledge of rationality is equivalent to assum-

ing that [both players know that]^n that both players are rational for all n. Hence, we let k go to infinity and players who know that [both players know that]$^\infty$ both players are rational choose strategies in the interval $[\lim_{k\to\infty} \underline{s}_k, \lim_{k\to\infty} \bar{s}_k]$, and

$$\underline{s}_k = \bar{s}_k = \frac{2\alpha m}{3m + 2\beta}$$

Hence, there is unique conjecture i could have on the play of j which is consistent with common knowledge of rationality.

In any game, a rational player is not going to play a strategy which is never best response. Trying to verify that there is no conjecture which would justify the strategy could be cumbersome, and therefore a natural question is whether we could characterize the set of justifiable actions with no reference to conjectures and expected payoff maximization. In the lecture notes, this question was answered positively for a general game - it was proven that a strategy, s_i, is never best reply (not justifiable) if and only if it is strictly dominated. Problem 3 provides intuition for the proof presented in the lecture notes with the particular example.

Problem 3 (Intuition for the separating hyperplane proof)**.**

Consider the following static game:

1. *Set of players, $\mathfrak{I} = \{1, 2\}$*

2. *Pure strategy space for player 1, $S_1 = \{a, b, c, d\}$*

3. *Pure strategy space for player 2, $S_2 = \{l, r\}$*

4. *von Neumann-Morgenstern utility function of player i, $u_i : S_1 \times S_2 \to \mathbb{R}$. Player 1's von Neumann-Morgenstern utility function, $u^1 : S_1 \times S_2 \to \mathbb{R}$ is presented in the matrix below*

Any conjecture player 1 might have on the play of 2 induces preference relation on the space of payoff pairs. Plot the graph where (i) horizontal axis represents payoff to player 1 when 2 plays l; (ii) vertical axis represents payoff to player 1 when 2 plays r.

	l	r
a	5	1
b	1	5
c	2	2
d	1	1

a) Plot the extreme points of the set of feasible expected payoff vectors of player 1 which is defined below

$$\mathbf{E} := \{(x, y) : \exists s_1 \in S_1 : x = u(s_1, l), y = u(s_1, r)\}$$

(Definition in words: the set of extreme points is the set of (x, y)-pairs such that player 1 has a strategy such that when this strategy is played and 2 plays l, 1's payoff is x, and such that when this strategy is played and 2 plays r, 1's payoff is y

b) Plot the set of feasible payoff vectors for player 1 (in other words, the set of all convex combinations of the vectors plotted in part *a*))

c) In the set of feasible payoff vectors, mark the payoff vectors which correspond to the strategies which are not dominated

d) Pick any payoff vector corresponding to the strategy which is not dominated, and plot the set of vectors in $\mathbb{R}^2$ which are strictly greater than the chosen payoff vector [6]

e) Plot a separating hyperplane - a line separating the set of feasible payoffs from the set plotted in part *d*) of the problem

f) Plot the vector normal to the separating hyperplane

g) Divide each coordinate of the normal vector by its norm

h) Show that the chosen strategy is the best-response to the conjecture corresponding to the normalized normal vector

As we saw in the Cournot competition example of the lecture notes or in problem 2 above, sometimes all but one conjecture on the play of others

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[6]Strictly greater: =greater in each coordinate

contradict assumption of common knowledge of rationality between the players in the game - and in such cases, we can make precise prediction on the outcome of the play. However this is not the case in most of the interesting games about which we still want to be able to say something of essence. In those games, a natural way to fix conjectures of the players about the play of others is correlated equilibrium demonstrated in problem 4 below.

Problem 4 (Correlated equilibrium)**.**

Consider the following two-player game:

	l	r
t	$5, 1$	$0, 0$
b	$4, 4$	$1, 5$

a) What is the set of rationalizable strategies in the game above?

Solution. For each player, the set of rationalizable strategies is the set of all the strategies she has available. To see this, plot the similar graph to the one we plotted in the previous problem and notice that all the strategies are on the payoff frontier and therefore are best-responses to some conjecture.

b) Find all Nash equilibria of the game. What is the best payoff players can get in the symmetric equilibrium?[7]

Solution. There are three Nash equilibria in this game: (t,l),(b,r) and $(\frac{1}{2}, \frac{1}{2})$. The best symmetric equilibrium is the mixed strategy equilibrium, since the pure strategy equilibria are not symmetric. It yields an expected payoff of $\frac{5+4+1}{4} = 2.5$.

c) Suppose that before choosing their actions, the players first toss a coin. After publicly observing the outcome of the coin toss, they choose simultaneously their action. Draw the extensive form of the described game and define available strategies for the players. Find a symmetric Nash equilibrium that gives both players a higher payoff than the symmetric equilibrium in *a)*.

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[7]In the symmetric equilibrium expected payoffs of the players must be the same

Solution. Players can now condition their actions on the coin toss. We can take this into account by for example denoting the strategies by double letters (tt), where the first one stands for the action after heads and the latter one after tails. The set of strategies are thus $S_1 = \{\text{tt}, \text{tb}, \text{bt}, \text{bb}\}$ and $S_2 = \{\text{ll}, \text{lr}, \text{rl}, \text{rr}\}$. The extensive form will be drawn in class, and the strategic form is presented in table below:

	ll	lr	rl	rr
tt	$5, 1$	$2.5, \frac{1}{2}$	$2.5, \frac{1}{2}$	$0, 0$
tb	$4.5, 2.5$	$3, 3$	$2, 2$	$\frac{1}{2}, 2.5$
bt	$4.5, 2.5$	$2, 2$	$3, 3$	$\frac{1}{2}, 2.5$
bb	$4, 4$	$2.5, 4.5$	$2.5, 4.5$	$1, 5$

Table 2: Strategic form of the game with the coin toss

Now, it is easy to check in the matrix above that (tb, lr) is a symmetric equilibrium with the payoffs of $(3, 3)$ which is larger than the symmetric equilibrium payoffs in b).

d) Suppose that there is a mediator that can make a recommendation separately and covertly for each player. Suppose that the mediator makes recommendation (t, l), or (b, l), or (b, r), each with probability $\frac{1}{3}$. Each player only observes her own action choice recommendation (so that, e.g., the row player upon seeing the recommendation b does not know whether the recommended profile is (b, l) or (b, r). Does any of the players have an incentive to deviate from the recommended action? What is the expected payoff under this scheme?

Solution. Recommendations are a neat way to fix the conjecture of the player about the action of the others. To see this and to confirm that conforming to recommendations constitutes an equilibrium, lets go through the recommended strategies case by case:

Player 1
Recommendation: $\text{t} \Rightarrow$ player 1 knows that player 2 will play l. No incentive to deviate, since the payoff from t is higher than from b ($5 > 4$).
Recommendation: $\text{b} \Rightarrow$ player 2 will play l or r with equal probability. Player 1 is indifferent between t and b: $\frac{1}{2} \cdot 5 + 0 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$. So no

incentive to deviate.

Player 2

Recommendation: $l \Rightarrow$ player 2 knows that player 1 will play t and b with equal probability. Player 2 is indifferent between l and r:
$\frac{1}{2} \cdot 4 + 1 \cdot \frac{1}{2} = 5 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}$.

Recommendation: $r \Rightarrow$ player 2 knows that player 1 will play d. Payoff from r is strictly higher than from l $(5 > 4)$.

The players, therefore, have no incentive to deviate. The expected payoff from the scheme is $\frac{1+4+5}{3} = 3\frac{1}{3}$. This is because (D,L), which offers the highest sum of payoffs, is now played with probability $\frac{1}{3}$.

Problem 5 (Zero-sum game).

Consider the following two-player zero-sum game.

	l	c	r
t	3	-3	0
m	2	6	4
b	2	5	6

Table 3: Payments player 2 makes to player 1

a) Find a mixed strategy of player 1 that guarantees him the same payoff against any pure strategy of player 2.

Solution. We need to solve the following matrix equation:

$$\begin{bmatrix} 3 & 2 & 2 & -1 \\ -3 & 6 & 5 & -1 \\ 0 & 4 & 6 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving it, we get

$$\begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 0 \\ \frac{12}{5} \end{bmatrix}$$

b) Find a mixed strategy of player 2 that guarantees him the same payoff against any pure strategy of player 1.

We need to solve the following matrix equation:

$$\begin{bmatrix} -3 & 3 & 0 & -1 \\ -2 & -6 & -4 & -1 \\ -2 & -5 & -6 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving it, we get

$$\begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} \frac{22}{25} \\ \frac{2}{5} \\ \frac{1}{5} \\ -\frac{12}{5} \end{bmatrix}$$

c) Does the play of the mixed strategies found above constitutes Nash equilibrium of the game?

Lemma 1. *Strategies found in* a) *and* b) *constitute Nash equilibrium of the game.*

Proof. Fix a candidate Nash equilibrium of the game, and suppose that it features strategies we found in a) and b). Nash equilibrium is a solution concept assuming that players make correct conjectures about their opponents' actions (either mixed or pure), and best respond to the correct conjecture. So if we want to verify that the strategies indeed constitute NE (Nash equilibrium) of the game, we need to make sure that each player best responds to the correct conjecture given by his opponent's true strategy.

Then, fix the strategy of player 2 to be the one we found in b) and assume that 1 makes the correct conjecture that 2 is using the strategy from part b). Then, by construction of the strategy in b), 1 is indifferent between all his strategies, meaning that any mixed strategy would constitute best-reponse. Thus, 1's strategy is indeed best-response to the correct conjecture.

Then, fix the strategy of player 1 to be the one we found in a) and assume that 2 makes the correct conjecture that 1 is using the strategy from part a). Then, again by construction 2 is indifferent between all his strategies, so he might as well mix and that would be best-response.

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Notice a general unappelaing feature of mixed strategies - fixing a conjecture on the actions of player 2 (1), player 1 (2) does *not* need to mix to best-respond to the conjecture - playing any of the pure strategies in the support of the mixed strategy would be as good of a best response as the mixed strategy itself. Why to mix then? Then the only reason for 1 (2) to mix is not to increase the own utility, but to sustain stability captured by the equilibrium concept.

References

1. P. Battigalli, *Game Theory: Analysis of Strategic Thinking*, lecture notes, 2017.